1. *(**10pts**) Given the equation

$$\Big(3x + \frac{6}{y}\Big) + \Big(\frac{x^2}{y} + \frac{3y}{x}\Big)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

   (a) Show that this is not an exact equation,

   (b) Determine the values of the constants α and β, such that $\mu(x, y) = x^\alpha y^\beta$ is an integrating factor for this equation;

   (c) By using the integral factor found above, derive the general solution of the equation.

Solution: Let

$$\widetilde{M}(x, y) = x^\alpha y^\beta \Big(3x + \frac{6}{y}\Big), \quad \widetilde{N}(x, y) = x^\alpha y^\beta \Big(\frac{x^2}{y} + \frac{3y}{x}\Big).$$

Then

$$\frac{\partial \widetilde{M}}{\partial y} = 3\beta x^{\alpha+1} y^{\beta-1} + 6(\beta - 1)x^\alpha y^{\beta-2},$$
$$\frac{\partial \widetilde{N}}{\partial x} = (\alpha + 2)x^{\alpha+1} y^{\beta-1} + 3(\alpha - 1)x^{\alpha-2} y^{\beta+1}$$

In order to have

$$\frac{\partial \widetilde{M}}{\partial y} = \frac{\partial \widetilde{N}}{\partial x},$$

we derive that $3\beta = \alpha + 2$, and

$$\alpha = \beta = 1.$$

Therefore, we derive $\mu(x, y) = xy$,

$$\widetilde{M}(x, y) = 3x^2 y + 6x, \quad \widetilde{N}(x, y) = x^3 + 3y^2.$$

$$\frac{\partial F}{\partial x} = 3x^2 y + 6x, \quad F = x^3 y + 3x^2 + h(y).$$

From

$$\frac{\partial F}{\partial y} = x^3 + h'(y) = x^3 + 3y^2,$$

we obtain $h(y) = y^3$. Therefore, $F(x,y) = x^3y + 3x^2 + y^3$, and The solution is

$$F(x,y) = x^3y + 3x^2 + y^3 = C.$$

2. ***(5pts)** Perform the phase line analysis for the following autonomous equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3y^2(2y-1)(y-2),$$

and determine that

- its equilibrium states;
- the type of each equilibrium state,
- the stability property of each equilibrium state,
- sketch the integral curves in the physical plane (t, y), based on the above phase line analysis without solving the equation.

Solution:

- its equilibrium states: 1.$y = 0$, 2.$y = \frac{1}{2}$, 3.$y = 2$;
- the type of each equilibrium state: 1. Node; 2. Sink; 3. Source.
- the stability property of each equilibrium state: 1. Semi-stable; 2. Stable, 3. Unstable;
- sketch the integral curves in the physical plane (t, y), based on the above phase line analysis without solving the equation.

3. ***(15pts)** Given the equation:

$$(D^3 - D^2 - D + 1)y = 2\mathrm{e}^{-t} + 3.$$

(a) Find the annihilator for its inhomogeneous term;

(b) Derive a particular solution for the equation;

(c) Derive the general solution of the equation;

(d) Find the solution of the equation subject to the initial conditions:

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1.$$

Solution:

(a) $Q(D) = D(D+1)$

(b) $Q(D)P(D)y = D(D-1)^2(D+1)^2 y = 0.$

$$y_H = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + c_3 t\mathrm{e}^{t}.$$

and

$$y_P = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + c_3 t\mathrm{e}^{t} + d_1 + d_2 t\mathrm{e}^{-t}.$$

We may only consider:

$$y_P = d_1 + d_2 t\mathrm{e}^{-t}.$$

Substitute it into the equation

$$(D^3 - D^2 - D + 1)y_P = 2\mathrm{e}^{-t} + 3,$$

it follows that

$$d_1 = 3, \quad d_2 = \frac{1}{2}.$$

(c) The general solution is

$$y_P = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + c_3 t\mathrm{e}^{t} + 3 + \frac{1}{2} t\mathrm{e}^{-t}.$$

(d) From the IC's, we may derive c_1, c_2, c_3, so that

$$y_P = 3 + \frac{1}{2} t\mathrm{e}^{-t} - \frac{9}{4} c_1 \mathrm{e}^{-t} + \frac{1}{4} \mathrm{e}^{t} + 2t\mathrm{e}^{t}.$$

4. ***(10pts)** Given the equation in the interval $(x > 0)$:

$$x^2 y'' - 2xy' + 2y = 2x^4$$

(a) Derive a particular solution of the equation with the method of variation of parameter,

(b) Write down the general solution of the given inhomogeneous equation,

(c) Find the solution of the equation which satisfies the initial conditions:

$$y(1) = 1, \quad y'(1) = 1.$$

Solution:

(a) Derive a particular solution of the equation with the method of variation of parameter: The two linear independent solutions of associated equation are: Let $x = \mathrm{e}^t$,

$$[D(D-1) - 2D + 2]\tilde{y}_H = (D-1)(D-2)y_H = 0.$$

$$y_H = c_1\mathrm{e}^{2t} + c_2\mathrm{e}^t = c_1x^2 + c_2x = c_1y_1 + c_2y_2.$$

Let the particular solution,

$$y_P = c_1(x)y_1 + c_2(x)y_2$$

we derive that

$$\begin{aligned} c_1'(x)y_1 + c_2'(x)y_2 &= 0 \\ c_1'(x)y_1' + c_2'(x)y_2' &= 2x^2. \end{aligned}$$

It follows that

$$c_1'(x) = 2x, \quad c_2'(x) = -2x^2.$$

So that,

$$y_P(x) = \frac{1}{3}x^4.$$

(b) Write down the general solution of the given inhomogeneous equation:

$$y(x) = \frac{1}{3}x^4 + d_1x^2 + d_2x.$$

(c) Find the solution of the equation which satisfies the initial conditions:

$$y(1) = 1, \quad y'(1) = 1.$$

It follows that $d_1 = -1, d_2 = f53$.

$$y(x) = \frac{1}{3}x^4 - x^2 + \frac{5}{3}x.$$

5. ***(10pts)** Given the following equation

$$x(x-1)y'' + 2xy' + 6\mathrm{e}^x y = 0$$

(a) Find all the regular singular points

(b) Derive the indicial equation and the exponents at the singularity for each regular singular point;

(c) Write down the forms of the series of two linear independent solutions near the smallest regular singular point $\{y_1(x), y_2(x)\}$. You do not need to calculate the coefficients of the series.

(d) the equation has no non-zero solution bounded near the regular singular point.

Solution:

(a) Find all the regular singular points: $x = 0, x_1$.

(b) Derive the indicial equation and the exponents at the singularity for each regular singular point:
For the RSP $x = 0$, $F(r) = r^2 - r = 0, \quad r_1 = 1, r_2 = 0.$
For the RSP $x = 1$, $F(r) = r^2 + r = 0, \quad r_1 = 0, r_2 = -1.$

(c) Write down the forms of the series of two linear independent solutions near the smallest regular singular point $\{y_1(x), y_2(x)\}$. Around $x = 0$, the difference of two roots is an integer, so that

$$y_1(x) = |x| \sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = a y_1(x) \ln |x| + 1 + \sum_{n=1} c_n x^n.$$

(d) Determine which of the following cases is true without solving the equation near the smallest regular singular point:

- the equation has all solutions bounded near the regular singular point, **true**

6. ***(10pts)**

(a) Sketch the following functions in the interval $t > 0$ and find the Laplace transform of these functions:

$$f(t) = (t-3)u_2(t) - (t-3)u_3(t);$$

Solution:

$$F(s) = -\frac{\mathrm{e}^{-2s}}{s} + \frac{1}{s^2}\left[\mathrm{e}^{-2s} - \mathrm{e}^{-3s}\right]$$

(b) Find the inverse Laplace transform of the following function:

$$F(s) = \frac{2s+1}{4s^2+4s+5},$$

Solution:

$$F(s) = \frac{1}{2}\frac{s+1/2}{(s+1/2)^2+1},$$

Hence,

$$f(t) = \frac{1}{2}\mathrm{e}^{-t/2}\cos t + \cdots .$$

7. ***(10pts)** Solve the following IVP's with the Laplace transform method:

$$y'' - 2y' + 2y = \mathrm{e}^{-t}, \quad y(0) = 0,\ y'(0) = 1.$$

Solution:

$$Y(s)[s^2 - 2s + 2] - 1 = \frac{1}{s+1}.$$

so,

$$\begin{aligned} Y(s) &= \tfrac{1}{[s^2-2s+2]} + \tfrac{1}{(s+1)[s^2-2s+2]} \\ &= \tfrac{1}{5}\tfrac{1}{s+1} + \tfrac{1}{(s-1)^2+1} + \tfrac{3/5-s/5}{(s-1)^2+1} \end{aligned}$$

It follows That

$$y(t) = \frac{1}{5}\mathrm{e}^{-t} + \frac{7}{5}\mathrm{e}^{t}\sin t - \frac{1}{5}\mathrm{e}^{t}\cos t.$$